§3. Renormalization of Gauge Theories

Last time: symmetry of eff. action

$$0 = \int d^4y \, \langle F^n(y) \rangle_{J_X} \frac{\delta \Gamma[X]}{\delta X^n(y)}$$

where $F^n(y)$ is generator of symmetry of original action:

$$X^n(y) \longmapsto X^n(y) + \varepsilon F^n[y; X]$$

Now we apply this to BRST invariance of action $I[X]$ for non-abelian gauge theories:

$$\int d^4x \, \langle \Delta^n(x) \rangle_{J_X} \frac{\delta_L \Gamma[X]}{\delta X^n(x)} = 0 \qquad (1)$$

where the change in $X^n(x)$ under a BRST transformation with infitesimal fermionic parameter θ is

$$\delta_\theta X^n(x) = \theta \Delta^n(x)$$

and $\langle \cdots \rangle$ denotes the vacuum expectation value in the presence of a current J_X.

$\longrightarrow$ since BRST trf. is <u>non-linear</u>, eq. (1) does <u>not</u> tell us that eff. action is invariant under it.

$\longrightarrow$ introduce modified effective action

$$\Gamma[\chi, K] \equiv W[\mathcal{J}_{\chi,K}, K] - \int d^4x\, \chi^n(x)\, \mathcal{J}_{\chi,K\,n}(x)$$

where W is here calculated with gauge-fixed action $I + \int d^4x\, \Delta^n K_n$:

$$e^{iW[\mathcal{J},K]} \equiv \int \left[\prod_{n,x} d\chi^n(x)\right] \exp\left(iI + i\int d^4x\, \Delta^n K_n + i\int d^4x\, \chi^n \mathcal{J}_n\right)$$

and $\mathcal{J}_{\chi,K}$ is the current satisfying:

$$\left.\frac{\delta_R W[\mathcal{J},K]}{\delta \mathcal{J}_n(x)}\right|_{\mathcal{J}=\mathcal{J}_{\chi,K}} \equiv \chi^n(x). \qquad (2)$$

K_n here have the same fermionic/bosonic statistics as Δ^n, which is opposite to χ^n.

$\longrightarrow$ since $\Delta^n(x)$ are BRST-invariant (BRST trf. is nilpotent), we get

$$\int d^4x\, \langle \Delta^n(x) \rangle_{\mathcal{J},K}\, \frac{\delta_L \Gamma[\chi, K]}{\delta \chi^n(x)} = 0,$$

where $\langle \cdots \rangle_{\mathcal{J},K}$ denotes a vacuum expectation value in the presence of the current $\mathcal{J}$ and the external fields K:

$$\langle \mathcal{O}[\chi] \rangle_{\mathcal{J},K} = \frac{\int \left[\prod_{n,x} d\chi^n(x)\right] \mathcal{O}(\chi) \exp\left(iI + i\int d^4x\, \Delta^n K_n + i\int d^4x\, \chi^n \mathcal{J}_n\right)}{\int \left[\prod_{n,x} d\chi^n(x)\right] \exp\left(iI + i\int d^4x\, \Delta^n K_n + i\int d^4x\, \chi^n \mathcal{J}_n\right)} \qquad (3)$$

We can reexpress $\langle \Delta^n(x) \rangle_{J,K}$ as follows:

$$\frac{\delta_R \Gamma[\chi, K]}{\delta K_n(x)} = \left.\frac{\delta_R W[J,K]}{\delta K_n(x)}\right|_{J=J_{\chi K}} + \int d^4y \left.\frac{\delta_R W[J,K]}{\delta J_m(y)}\right|_{J=J_{\chi,K}} \frac{\delta_R J_{\chi,K\,m}(y)}{\delta K_n(x)}$$

$$- \int d^4y\, \chi^m(y) \frac{\delta_R J_{\chi,K\,m}(y)}{\delta K_n(x)}$$

Using equation (2) and (3), we get

$$\frac{\delta_R \Gamma[\chi, K]}{\delta K_n(x)} = \left.\frac{\delta_R W[J,K]}{\delta K_n(x)}\right|_{J=J_{\chi,K}} = \langle \Delta^n(x) \rangle_{J,K} .$$

Now equation (1) becomes

$$\int d^4x \frac{\delta_R \Gamma[\chi, K]}{\delta K_n(x)} \frac{\delta_L \Gamma[\chi, K]}{\delta \chi^n(x)} = 0$$

"Zinn-Justin equation"

As exchange of fields and anti-fields leads to a minus sign,
for examample $\frac{\delta_L(\omega_1 \omega_2)}{\delta \omega_1} = - \frac{\delta_R(\omega_1\omega_2)}{\delta\omega_1}$ for ω_1 and ω_2 fermionic, this can be rewritten as

$$(\Gamma, \Gamma) = 0 \quad , \quad \text{where} \qquad (4)$$

$$(F, G) = \int d^4x \frac{\delta_R F[\chi, K]}{\delta \chi^n(x)} \frac{\delta_L G[\chi, K]}{\delta K_n(x)} - \int d^4x \frac{\delta_R F[\chi, K]}{\delta K_n(x)} \frac{\delta_L G[\chi, K]}{\delta \chi^n(x)}$$

Question: What is the form of the effective action $\Gamma[x, K]\big|_{K=0}$?

$\Gamma[x, K]$ is complicated functional of both x and K

$\rightarrow$ write the action $S[x, K] = I[x] + \int d^4x \, \Delta^n K_n$

as $S[x, K] = S_R[x, K] + S_\infty[x, K]$

renormalized action

contains counterterms to cancel infinities from loop graphs

$\rightarrow$ both S_R and S_∞ must have symmetries of original action

$\rightarrow$ Do the infinite parts of Γ share the same symmetries ?

To answer this question, we expand Γ in loop order:

$$\Gamma[x, K] = \sum_{N=0}^{\infty} \Gamma_N[x, K] .$$

$\rightarrow$ eq. (4) becomes at N loop order

$$\sum_{N'=0}^{N} (\Gamma_{N'}, \Gamma_{N-N'}) = 0 \qquad (5)$$

Leading term of (5) is $\Gamma_0[x, K] = S_R[x, K] \rightarrow$ finite

Proceed now recursively:

Suppose that, for all $M \leq N-1$, all infinities for M-loop graphs have been cancelled by counterterms in S_∞.

$\rightarrow$ infinities can appear only in the $N' = 0$ and $N' = N$ terms:

$$\left(S_R, \underset{\substack{\uparrow \\ \text{infinite part of } \Gamma_N}}{\Gamma_{N,\infty}}\right) = 0 \qquad (6)$$

- $\Gamma_{N,\infty}[\chi, K]$ can only be a sum of products of fields and their derivatives
  $\rightarrow$ total dimensionality of these products must be 4
- $\Gamma_{N,\infty}[\chi, K]$ is invariant under all "linearly" realized symmetry trfs. of $I[\chi]$ (see previous lecture)

$\rightarrow$ want to know the dimensionality of fields K_n.

if $\dim(\chi^n) = d_n \rightarrow \dim(\Delta^n) = d_n + 1$

$\rightarrow$ from $\dim\left(\int d^4x \, K_n \Delta^n\right) = 0$ we get $\dim(K_n) = 3 - d_n$

The fields $A^{\alpha\mu}$, ω^α, and $\omega^{\alpha *}$ all have dimensionalities $+1 \longrightarrow \dim(K_n) = +2$

Spin $1/2$ matter fields ψ_ℓ have dim $3/2$

$\longrightarrow \dim(K_n) = 3/2$

Thus $\Gamma_{N,\infty}[\chi, K]$ is at most quadratic in K_n

Let us now come to ghost numbers:

If χ^n has ghost number $\gamma_n + 1$

$\longrightarrow \mathrm{ghost}(K_n) = -\gamma_n - 1$

We have:

- $\mathrm{ghost}(A^{\alpha,\mu}) = 0 \Rightarrow \mathrm{ghost}(K_A) = -1$
- $\mathrm{ghost}(\psi^\ell) = 0 \Rightarrow \mathrm{ghost}(K_\psi) = -1$
- $\mathrm{ghost}(\omega^\alpha) = +1 \Rightarrow \mathrm{ghost}(K_\omega) = -2$
- $\mathrm{ghost}(\omega^{\alpha *}) = -1 \Rightarrow \mathrm{ghost}(K_{\omega^*}) = 0$

Since $\mathrm{ghost}(\Gamma_{N,\infty}[\chi, K]) = 0$, we have only <u>at most linear</u> terms in K_n.

(Also linear in K_α^*: $\dfrac{\delta_L \Gamma_{N,\infty}[\chi, K]}{\delta K_\alpha^*} = \langle \Delta^{\alpha *} \rangle_{\mathcal{J},K} = -h^\alpha$

since $\Delta^{\alpha *} = -h^\alpha$ and independent of all K_n)

$$\longrightarrow \Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi, 0] + \int d^4x\, \mathcal{D}_N^n[\chi; x] K_n(x)$$

Recall that

$$S_R[\chi, K] = S_R[\chi] + \int d^4x\, \Delta^\mu[\chi; x] K_\mu(x)$$

Thus equation (6) becomes

$$(7)\quad \int d^4x \left[\Delta^\mu[\chi;x] \frac{\delta_L \Gamma_{N,\infty}[\chi,0]}{\delta \chi^\mu(x)} + \mathcal{D}_N^\mu[\chi;x] \frac{\delta_L S_R[\chi]}{\delta \chi^\mu(x)} \right] = 0,$$

and

$$(8)\quad \int d^4x \left[\Delta^\mu(\chi;x) \frac{\delta_L \mathcal{D}_N^m(\chi;y)}{\delta \chi^\mu(x)} + \mathcal{D}_N^\mu(\chi;x) \frac{\delta_L \Delta^m(\chi;y)}{\delta \chi^\mu(x)} \right] = 0$$

These equations can be put into a nice form by defining

$$\Gamma_N^{(\varepsilon)}[\chi] \equiv S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi,0],$$

and $\Delta_N^{(\varepsilon)\mu}(x) \equiv \Delta^\mu(x) + \varepsilon \mathcal{D}_N^\mu(x),$

with ε infinitesimal.

Then eq. (7) just says that $\Gamma_N^{(\varepsilon)}[\chi]$ is invariant under the trf.

$$\chi^\mu(x) \longmapsto \chi^\mu(x) + \theta \Delta_N^{(\varepsilon)\mu}(x), \qquad (*)$$

and eq. (8) implies that (*) is "nilpotent" (using invariance under original BRST trf.)